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RESONANCES AND ASYMPTOTIC TRAJECTORIES IN HAMILTONIAN SYSTEMS*

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The existence of motions asymptotic to the equilibrium state of a Hamiltonian system with an arbitrary finite number of degrees of freedom is investigated. It is assumed that the Hamiltonian function is analytical in the neighbourhood of the equilibrium and is either time-periodic or time-independent. The characteristic exponents of the linearized equations of motion are purely imaginary and a simple third- or fourth-order resonance is observed. The sufficient conditions for asymptotic motions to exist are derived, and their approximate analytical representation is constructed in a fairly small neighbourhood of the position of equilibrium.

Assume that the motion of a system with n degrees of freedom is described by the following canonical differential equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \quad (j = 1, 2, \ldots, n) \tag{1}$$

and $q_j = p_j = 0$ is a position of equilibrium. The solution $q_j = f_j(t)$, $p_j = g_j(t)$ of Eqs.(1) that does not vanish identically is said to be asymptotic to the solution $q_j = p_j = 0$ if $\lim f_j(t) = \lim g_j(t) = 0$ as $t \to +\infty$ or $t \to -\infty$. In the first case the solution is called of type a_+ and in the second case of type a_-.

A well-known classical algorithm /1, 2/ provides sufficient conditions for the solutions a_+ and a_- to exist and generates them in the form of series. One of the main conditions for the algorithm to be applicable is that the linearized system of Eqs.(1) has at least one non-zero characteristic value. In Hamiltonian systems, characteristic values exist in pairs $\pm\varkappa_j$ $(j = 1, 2, \ldots, n)$, and therefore the theory of Lyapunov and Poincaré is applicable to (1) only if the equilibrium is unstable in the first (linear) approximation. In what follows we will assume that the equilibrium is stable to a first approximation. The Hamiltonian H is assumed to be either 2π-periodic in t or time-independent in a sufficiently small neighbourhood of the point $q_j = p_j = 0$.

Asymptotic trajectories of conserative systems were studied in /3-5/ in connection with the inversion of the Lagrange theorem of stability of equilibria. Some results of these studies were extended in /6/ to non-natural systems. Asymptotic motions for Hamiltonian systems with one degree of freedom and a 2π-periodic Hamiltonian were studied in /7, 8/ for the case of zero characteristic values; motions asymptotic to stable equilibria in the linear approximation for an anutonomous Hamiltonian system with two degrees of freedom were studied in /9/. The trajectories asymptotic to the periodic trajectories of an autonomous Hamiltonian system with two degree of freedom were studied in /10/.

In this paper we consider the existence and analytical struture of solutions asymptotic to the equilibrium $q_j = p_j = 0$ of system (1) for an arbitrary number of degrees of freedom n. We assume that the characteristic exponents $\pm i\lambda_j$ $(j = 1, 2, \ldots, n)$ of the linearized system are purely imaginary and there are no resonances to second order inclusive, i.e., the equality

$$k_1\lambda_1 + k_2\lambda_2 + \ldots + k_n\lambda_n = N, \tag{2}$$

where N is an integer ($N = 0$ if H is time-independent), cannot hold for integer k_j, the sum of the moduli of which is 1 or 2.

With appropriately chosen variables q_j, p_j, the Hamiltonian function in the neighbourhood of the point $q_j = p_j = 0$ can be represented in series form

$$H = \frac{1}{2}\sum_{j=1}^{n} \lambda_j (q_j^2 + p_j^2) + \ldots \tag{3}$$

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where ellipsis denotes the collection of monomials of higher than second degree in q_j, p_j ($j = 1, 2, \ldots, n$) with 2π-periodic coefficients.

We will consider simple third- and fourth-order resonances, when equality (2) is satisfied only for one combination of non-negative integers k_j which sum to 3 or 4.

With third- or fourth-order resonances, we can apply a nearly identical real change of variables $q_j, p_j \to \xi_j, \eta_j$ which is 2π-periodic in t and analytical in ξ_j, η_j to reduce the Hamiltonian (3) to the form /11/

$$H = \sum_{j=1}^{n} \lambda_j \rho_j + \sum_{i,j=1(i\leq j)}^{n} c_{ij}\rho_i\rho_j + \rho_1^{k_1/2}\rho_2^{k_2/2}\ldots\rho_n^{k_n/2}(\sigma\sin\theta + \delta\cos\theta) + \ldots$$
$$(\theta = k_1\theta_1 + k_2\theta_2 + \ldots + k_n\theta_n - Nt)$$

where c_{ij}, σ, and δ are constants, and the ellipsis denotes the collection of terms of higher than fourth degree in $\xi_j = \sqrt{2\rho_j}\sin\theta_j$, $\eta_j = \sqrt{2\rho_j}\cos\theta_j$ ($j = 1, 2, \ldots, n$) with 2π-periodic coefficients.

We make the canonical change of variables $\theta_j, \rho_j \to \varphi_j, r_j$:

$$\theta_j = \varphi_j + \lambda_j t + \theta_j^*, \quad \rho_j = \alpha r_j$$
$$(k_1\theta_1^* + k_2\theta_2^* + \ldots + k_n\theta_n^* = \theta^*, \quad \sin\theta^* = -\delta(\sigma^2+\delta^2)^{-1/2},$$
$$\cos\theta^* = \sigma(\sigma^2+\delta^2)^{-1/2})$$

Here $\alpha = (\sigma^2+\delta^2)^{-1}$ for third-order resonance and $\alpha = (\sigma^2+\delta^2)^{-1/2}$ for fourth-order resonance.

In the new variables, Eqs.(1) are rewritten in the form

$$\frac{d\varphi_j}{dt} = \frac{\partial H}{\partial r_j}, \quad \frac{dr_j}{dt} = -\frac{\partial H}{\partial \varphi_j} \quad (j = 1, 2, \ldots, n) \tag{4}$$

$$H = \sum_{i,j=1(i\leq j)}^{n} a_{ij} r_i r_j + r_1^{k_1/2} r_2^{k_2/2} \ldots r_n^{k_n/2}\sin\varphi + H^* \tag{5}$$
$$a_{ij} = \alpha c_{ij}, \quad \varphi = k_1\varphi_1 + k_2\varphi_2 + \ldots + k_n\varphi_n$$

H^* is the collection of terms of higher than fourth degree in $\sqrt{r_j}$ ($j = 1, 2, \ldots, n$).

Changing if necessary the indexing of λ_j, we may assume that the following relationships from (2) correspond to third-order resonance:

$$1)\ 3\lambda_1 = N, \quad 2)\ \lambda_1 + 2\lambda_2 = N, \quad 3)\ \lambda_1 + \lambda_2 + \lambda_3 = N \tag{6}$$

and the following relationships correspond to fourth-order resonance:

$$4)\ 4\lambda_1 = N, \quad 5)\ \lambda_1 + 3\lambda_2 = N, \quad 6)\ 2(\lambda_1 + \lambda_2) = N \tag{7}$$
$$7)\ \lambda_1 + \lambda_2 + 2\lambda_3 = N, \quad 8)\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = N$$

Let us first consider the approximate system (4), omitting in the Hamiltonian (5) terms of higher than third degree in $\sqrt{r_j}$ for resonances (6) and higher than fourth degree for resonances (7). Direct integration leads to the following results for asymptotic solutions of the approximate system.

1) $3\lambda_1 = N$. There exist three one-parameter families of solutions of type a_+ in which $\varphi_1 = i2\pi/3$ ($i = 0, 1, 2$), and $r_1(t) = 4r_1(0)(2 + 3\sqrt{r_1(0)}\,t)^{-2}$; $\varphi_j = 0$, $r_j = 0$ ($j \geq 2$), and three one-parameter families of solutions of type a_- in which $\varphi_1 = (2l+1)\pi/3$ ($l = 0, 1, 2$); $r_1(t) = 4r_1(0)(2 - 3\sqrt{r_1(0)}\,t)^{-2}$; $\varphi_j = 0$, $r_j = 0$ ($j \geq 2$).

2) $\lambda_1 + 2\lambda_2 = N$. There exist two-parameter families of asymptotic solutions described by the formulas

$$\varphi_i(t) = \varphi_i(0) \quad (i = 1, 2); \quad r_1(t) = {}^1/_2 r_2(t) = r_1(0)(1 \pm \sqrt{r_1(0)}\,t)^{-2}$$
$$\varphi_j = 0, \quad r_j = 0 \quad (j \geq 3); \quad \varphi_1(0) = -2\varphi_2(0) + {}^1/_2(-1 \pm 1)\pi +$$
$$2k\pi$$

(k is an integer).

Here and in what follows, the upper sign corresponds to solutions of type a_+ and the lower sign to solution of type a_-.

3) $\lambda_1 + \lambda_2 + \lambda_3 = N$. There exist three-parameter families of asymptotic solutions a_+ and

a_-:

$$\varphi_i(t) = \varphi_i(0) \quad (i = 1, 2, 3); \quad r_1(t) = r_2(t) = r_3(t) = 4r_1(0)(2 \pm \sqrt{r_1(0)}\, t)^{-2}$$

$$\varphi_j = 0, \quad r_j = 0 \quad (j \geqslant 4); \quad \varphi_1(0) = -\varphi_2(0) - \varphi_3(0) + {}^1/_2(-1 \pm 1)\pi + 2k\pi$$

(k is an integer).

4) $4\lambda_1 = N$. If $|a_{11}| < 1$, then there exist four one-parameter families of solutions of type a_+ in which $\varphi_1 = -\gamma/4 + i\pi/2$ $(i = 0, 1, 2, 3)$, $\gamma = \arcsin a_{11}$; $r_1(t) = r_1(0)(1 + 4\cos\gamma r_1(0)t)^{-1}$, $\varphi_j = (4\cos\gamma)^{-1} a_{1j} \ln(1 + 4\cos\gamma r_1(0)t)$, $r_j = 0$ $(j \geqslant 2)$, and four one-parameter families of solutions of type a_- in which $\varphi_1 = \gamma/4 + \pi/4 + l\pi/2$ $(l = 0, 1, 2, 3)$, while $r_1(t) = r_1(0)(1 - 4\cos\gamma r_1(0)t)^{-1}$, $\varphi_j = -(4\cos\gamma)^{-1} a_{1j} \ln(1 - 4\cos\gamma r_1(0)t)^{-1}$, $r_j = 0$ $(j \geqslant 2)$.

5) $\lambda_1 + 3\lambda_2 = N$. If $|a_{11} + 3a_{12} + 9a_{22}| < 3\sqrt{3}$, there exist two-parameter families of asymptotic solutions a_+ and a_-:

$$\varphi_i(t) = \pm(2\cos\gamma)^{-1}(\beta_i - \sin\gamma)\ln(1 \pm 3\sqrt{3}\cos\gamma\, r_1(0)t) + \varphi_i(0) \quad (i = 1, 2).$$

$$\gamma = \arcsin[(a_{11} + 3a_{12} + 9a_{22}) \mid (3\sqrt{3})], \quad \beta_1 = (2\sqrt{3} \mid 9)(2a_{11} + 3a_{12})$$

$$\beta_2 = (2\sqrt{3} \mid 9)(a_{12} + 6a_{22}); \quad r_1(t) = {}^1/_3 r_2(t) = r_1(0)(1 \pm 3\sqrt{3}\cos\gamma r_1(0)t)^{-1}$$

$$\varphi_j(t) = \pm(3\sqrt{3}\cos\gamma)^{-1}(a_{1j} + 3a_{2j})\ln(1 \pm 3\sqrt{3}\cos\gamma r_1(0)t), \quad r_j = 0 \quad (j \geqslant 3)$$

$$\varphi_1(0) = -3\varphi_2(0) \mp \gamma + {}^1/_2(-1 \pm 1)\pi + 2k\pi$$

(k is an integer).

6) $2(\lambda_1 + \lambda_2) = N$. If $|a_{11} + a_{12} + a_{22}| < 1$, there exist two-parameter families of asymptotic solutions a_+ and a_-:

$$\varphi_i(t) = \pm(2\cos\gamma)^{-1}(\beta_i - \sin\gamma)\ln(1 \pm 2\cos\gamma r_1(0)t) + \varphi_i(0) \quad (i = 1, 2)$$

$$\gamma = \arcsin(a_{11} + a_{12} + a_{22}), \quad \beta_1 = 2a_{11} + a_{12}, \quad \beta_2 = a_{12} + 2a_{22}$$

$$r_1(t) = r_2(t) = r_1(0)(1 \pm 2\cos\gamma r_1(0)t)^{-1}$$

$$\varphi_j(t) = \pm(2\cos\gamma)^{-1}(a_{1j} + a_{2j})\ln(1 \pm 2\cos\gamma r_1(0)t), \quad r_j = 0 \quad (j \geqslant 3)$$

$$\varphi_1(0) = -\varphi_2(0) \mp {}^1/_2\gamma + {}^1/_4(-1 \pm 1)\pi + k\pi$$

(k is an integer).

7) $\lambda_1 + \lambda_2 + 2\lambda_3 = N$. If $|a_{11} + a_{12} + 2a_{13} + a_{22} + 2a_{23} + 4a_{33}| < 2$, there exist three-parameter families of asymptotic solutions a_+ and a_-:

$$\varphi_i(t) = \pm(2\cos\gamma)^{-1}(\beta_i - \sin\gamma)\ln(1 \pm 2\cos\gamma r_1(0)t) + \varphi_i(0) \quad (i = 1, 2, 3)$$

$$\gamma = \arcsin[{}^1/_2(a_{11} + a_{12} + 2a_{13} + a_{22} + 2a_{23} + 4a_{33})], \quad \beta_1 = 2a_{11} + a_{12} + 2a_{13}$$

$$\beta_2 = a_{12} + 2a_{22} + 2a_{23}, \quad \beta_3 = a_{13} + a_{23} + 4a_{33}$$

$$r_1(t) = r_2(t) = {}^1/_2 r_3(t) = r_1(0)(1 \pm 2\cos\gamma r_1(0)t)^{-1}$$

$$\varphi_j(t) = \pm(2\cos\gamma)^{-1}(a_{1j} + a_{2j} + 2a_{3j})\ln(1 \pm 2\cos\gamma r_1(0)t), \quad r_j = 0 \quad (j \geqslant 4)$$

$$\varphi_1(0) = -\varphi_2(0) - 2\varphi_3(0) \pm \gamma + {}^1/_2(-1 \pm 1)\pi + 2k\pi$$

(k is an integer).

8) $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = N$. If $|a_{11} + a_{12} + a_{13} + a_{14} + a_{22} + a_{23} + a_{24} + a_{33} + a_{34} + a_{44}| < 1$, there exist four-parameter families of asymptotic solutions a_+ and a_-:

$$\varphi_i(t) = \pm(2\cos\gamma)^{-1}(2\beta_i - \sin\gamma)\ln(1 \pm \cos\gamma r_1(0)t) + \varphi_i(0) \quad (i = 1, 2, 3, 4)$$

$$\gamma = \arcsin(a_{11} + a_{12} + a_{13} + a_{14} + a_{22} + a_{23} + a_{24} + a_{33} + a_{34} + a_{44})$$

$$\beta_1 = 2a_{11} + a_{12} + a_{13} + a_{14}, \quad \beta_2 = a_{12} + 2a_{22} + a_{23} + a_{24}$$

$$\beta_3 = a_{13} + a_{23} + 2a_{33} + a_{34}, \quad \beta_4 = a_{14} + a_{24} + a_{34} + 2a_{44}$$

$$r_1(t) = r_2(t) = r_3(t) = r_4(t) = r_1(0)(1 \pm \cos\gamma r_1(0)t)^{-1}$$

$$\varphi_j(t) = \pm(\cos\gamma)^{-1}(a_{1j} + a_{2j} + a_{3j} + a_{4j})\ln(1 \pm \cos\gamma r_1(0)\, t), \quad r_j = 0$$
$$(j \geqslant 5)$$
$$\varphi_1(0) = -\varphi_2(0) - \varphi_3(0) - \varphi_4(0) \mp \gamma + {}^1/_2(-1 \pm 1)\pi + 2k\pi$$

(k is an integer).

These formulas provide an approximate representation of the asymptotic solutions of the complete (not the approximate) system of Eqs.(4) in a sufficiently small neighbourhood of the origin. Relying on the structure of approximate solutions and the known results on the representation of solutions of differential equations in the neighbourhood of a singular point /12/, we can prove the existence of asymptotic solutions of the complete system and obtain their analytical representation for large $|t|$.

As an example, consider the resonances 3 and 7, restricting the discussion to solutions of type a_+. Other asymptotic solutions for resonances (6) and (7) are considered similarly.

For resonance 3, we make the change of variables $r_k, \varphi_k, t \to x_k, y_k, \tau$ in system (4) using the formulas

$$r_i = \tau^2(4 + x_i), \quad \varphi_i = c_i + y_i \ (i = 1, 2, 3)$$
$$r_j = \tau^2 x_j, \quad \varphi_j = y_j \ (j \geqslant 4), \quad \tau = t^{-1} \tag{8}$$
$$(c_i \text{ is const}, \ c_1 = -c_2 - c_3 + 2k\pi, \ k = 0, \pm 1, \pm 2, \ldots)$$

In the new variables, system (4) is rewritten as

$$\tau dx_i/d\tau = -2x_i + x_1 + x_2 + x_3 + X_i \tag{9}$$
$$\tau dx_j/d\tau = -2x_j + X_j$$
$$\tau dy_i/d\tau = -y_1 - y_2 - y_3 - 4\sigma_i\tau + Y_i$$
$$\tau dy_j/d\tau = -4\sigma_j\tau + Y_j \ (i = 1, 2, 3; \ j \geqslant 4)$$
$$\sigma_1 = 2a_{11} + a_{12} + a_{13}, \quad \sigma_2 = a_{12} + 2a_{22} + a_{23}$$
$$\sigma_3 = a_{13} + a_{23} + 2a_{33}, \quad \sigma_j = a_{1j} + a_{2j} + a_{3j}$$

where the functions $X_k = X_k(\tau, x, y)$, $Y_k = Y_k(\tau, x, y)$ can be represented by the series

$$\sum_{m+m_1+\ldots+m_{2n}\geqslant 2} f_k^{(m, m_1, \ldots, m_{2n})}(\tau)\tau^m x_1^{m_1}\ldots x_n^{m_n} y_1^{m_{n+1}}\ldots y_n^{m_{2n}}$$

which converge in a sufficiently small neighbourhood of the point $x_k = y_k = 0$ if $|\tau| < \tau_1$, where τ_1 is a constant; the functions $f_k^{(m, m_1, \ldots, m_{2n})}(\tau)$ are real, continuous, and bounded.

Omitting the functions X_k, Y_k on the right-hand sides of system (9) and introducing a new independent variable $\eta = -\ln\tau$, we obtain an auxiliary system of linear differential equations with two positive characteristic values equal to 1. In accordance with the standard algorithm /12/, we can assert that system (9) has a one-parameter family of solutions which can be represented by the series

$$\sum_{m+m_1\geqslant 1} k^{(m, m_1)}(\tau)\tau^{m+m_1}c^{m_1} \tag{10}$$

which converge for $|\tau| < \tau_0$, $|c| < c_0$; τ_0 is a sufficiently small number, c is a constant parameter, and the functions $k^{(m, m_1)}(\tau)$ have the property $\lim k^{(m, m_1)}(\tau)\tau^\beta = 0$ as $\tau \to 0$ ($\beta = \text{const} > 0$).

In the variables r_k, φ_k, these solutions correspond to a three-parameter family of asymptotic solutions of type a_+ (with the parameters c_2, c_3, c). For sufficiently large t,

$$r_i = \frac{4}{t^2} + \frac{\psi_i}{t^3}, \quad r_j = \frac{\chi_j}{t^3}, \quad y_k = \frac{g_k}{t} \quad (i = 1, 2, 3; \ j \geqslant 4; \ k = 1, 2, \ldots)$$

where ψ_i, χ_j and g_k are functions of t, c_2, c_3, c uniformly bounded for sufficiently large t.

In the case of resonance 7, we make the change of variables $r_k, \varphi_k, t \to x_k, y_k, \tau$ in system (4) using the formulas

$$r_i = \frac{1}{t}\left(\frac{1}{2\cos\gamma} + x_i\right), \quad r_3 = \frac{1}{t}\left(\frac{1}{\cos\gamma} + x_3\right), \quad r_j = \frac{1}{t}x_j$$
$$\varphi_l = \frac{\beta_l - \sin\gamma}{2\cos\gamma}\ln t + c_l + y_l, \quad \varphi_j = \frac{a_{1j} + a_{2j} + 2a_{3j}}{2\cos\gamma}\ln t + y_j$$

$$\tau = t^{-1/4}\ (i = 1, 2;\ l = 1, 2, 3;\ j \geqslant 4)$$
$$(c_l \text{ is const, } c_1 = -c_2 - 2c_3 - \gamma + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots).$$

Rewriting system (4) in the new variables and transforming it (as in the case of resonance 3) into an auxiliary system of linear differential equations, we find that the latter has two positive characteristic values, 1 and 4. Therefore /12/, there exists a one-parameter family of solutions $x_k(\tau)$, $y_k(\tau)$ representable by convergent series similar to series (10) (τ^{m+m_1} is replaced by τ^{m+4m_1}). In the variables r_k, φ_k, these solutions correspond to a three-parameter family of asymptotic solutions of type a_+ (with the parameters c_2, c_3 and c).

Note that for resonance 7 (and also for other fourth-order resonances (7)), the order of decrease of x_k, y_k for large t is not less than $t^{-1/4}$ (unlike the third-order resonances (6), when x_k, y_k for large t are of order not less than t^{-1}).

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